Social Sciences
2023; 12(2): 47-59
http://www.sciencepublishinggroup.com/j/ss
doi: 10.11648/j.ss.20231202.11
ISSN: 2326-9863 (Print); ISSN: 2326-988X (Online)

Consecutively Halved Positional Voting: A Special Case of Geometric Voting

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To cite this article:

Peter Charles Mendenhall, Hal M. Switkay. Consecutively Halved Positional Voting: A Special Case of Geometric Voting. *Social Sciences*. Vol. 12, No. 2, 2023, pp. 47-59. doi: 10.11648/j.ss.20231202.11

Received: February 3, 2023; **Accepted**: February 21, 2023; **Published**: March 4, 2023

Abstract: The Borda count is a positional voting system that favors 'consensual' candidates with broad support while plurality is instead biased towards 'polarizing' ones with strong support. Our article focusses first on developing indices for quantifying system bias and then on vector analysis and design, while seeking to find an intermediate vector evenly balanced between consensus and polarization. The bias indices are based on the preference weightings of a normalized vector that represents a class of affine equivalent ones. The use of weightings that form a geometric progression evolves from this development. Such a 'geometric voting' vector can represent any positional voting vector with three preferences. With its common ratio as the sole variable, this vector can also span the whole spectrum of system bias continuously regardless of the number of preferences it employs; as demonstrated by our case study of the 1860 US presidential election with four candidates. Using this variable vector as an analytical tool, it establishes the 'consecutively halved positional voting' vector as the optimum one for balance. In our case study of the 2019 Nauru general election, this balanced vector is compared to its Dowdall rival that comprises a harmonic progression of weightings and several advantages are identified.

Keywords: Positional Voting, Consensus, Polarization, Geometric Progression, Borda Count, Plurality, Dowdall

1. Introduction

For a single-winner election, there is a wide variety of voting systems from which to choose. Even within the class of voting systems that rely on voters' preference ranking of candidates, there are several methods, including Instant Runoff Voting, Condorcet methods and positional voting. This third category deserves more attention, although Saari [1] has re-evaluated and promoted the Borda count.

The Borda count employs positional weightings that form an arithmetic sequence. The Dowdall method instead employs a harmonic sequence of weightings. We shall show that both these methods tend to favor consensus candidates, particularly when the number of candidates is large. We intensively study a third available mathematical sequence: a geometric one. As we vary the common ratio in the sequence between zero and one, we smoothly interpolate between plurality, the method most favorable to polarizing candidates, and the Borda count. Such common ratios do not appear to have been specifically addressed in social choice literature.

Our research seeks to investigate and compare these three types of positional voting sequences and how they affect the demotion or promotion of consensus and polarized candidates. We start from first principles in order to quantify system bias and then explore the full spectrum of vector bias from consensus to polarization. This analysis will answer the question of whether there are any intermediate non-arbitrary vectors that exhibit specific, critical, or useful properties.

In positional voting (PV), voters express their preferences for the various candidates (or options) by casting a ranked ballot. Each voter expresses a preference (P) for each candidate in strict descending rank order. For a PV election with a choice of $N \geq 2$ preferences, let i define the rank position from the top rank $i = 1$ for the highest preference P_1 down to the bottom rank $i = N$ for the lowest preference P_N. For the following analysis, every voter must express a preference of some rank for each candidate for their ballot to be valid. No truncation of preferences is permitted. Also, no two preferences cast by a voter may share the same rank. Tied preferences are invalid.

Each preference P_i awarded by a voter is given a weighting of value v_i according to its rank position i within the range $1 \leq i \leq N$. Let the vector $\mathrm{v} = (v_1, v_2, \ldots, v_{N-1}, v_N)$ represent the N weightings associated with each ballot. Any vector must satisfy two criteria [1]. Firstly, the weightings are non-increasing; namely, $v_i \geq v_{i+1}$ for all i such that $1 \leq i < N$. Secondly, the first preference must be worth more than the last preference; namely, $v_1 > v_N$.

Let V votes be the total number of valid ballots cast in a PV election. Each candidate is awarded a total of V preferences from the voters. The V weightings associated with these specific preferences are then totaled to become the tally (T_C) for that candidate (C). These tallies are then rank ordered from highest to lowest score. The collective rank ordering of the candidates is determined by their tallies. Since all candidates are fully ranked, one or more of the top ones may be elected. However, with the sole exception of the real-world case study in section 9, the focus throughout is on single-winner elections, so the candidate with the highest tally wins. Two or more candidates may tie in the resultant ranking. For practical elections, ties for first place may be resolved by employing a random tie-break.

Positive affine transformations of the weighting vectors do not change candidate rank orders. That is, if $m > 0$ and k is arbitrary, then $T_A > T_B$ implies $m(T_A + Vk) > m(T_B + Vk)$; as confirmed by Saari [1]. Therefore, different vectors may generate identical candidate rankings under all voter profiles. Such vectors are hence affine equivalent. It is useful to employ just one 'normalized' vector $\mathrm{w} = (w_1, \ldots, w_N)$ to represent all those vectors that are affine equivalent to it. The normalized vector used here is $\mathrm{w} = (1, \ldots, 0)$ where $w_1 = 1$ and $w_N = 0$. To normalize vector v, v_N is subtracted from each of its components and then each one is divided by $v_1 - v_N$. The components of vector w are hence determined using the equation $w_i = (v_i - v_N)/(v_1 - v_N)$ for affine equivalence; that is, $\mathrm{w} \equiv \mathrm{v}$.

No two distinct normalized PV vectors will produce the same collective candidate rankings under all voter profiles. The Borda count vector $(1, 1-d, 1-2d, \ldots, 2d, d, 0)$, where $d = 1/(N-1)$ is the common difference between consecutive weightings, generates very different outcomes to those of the plurality-equivalent vector $(1, 0, \ldots, 0)$. Candidates attracting both strong support and opposition may be characterized as 'polarizing' while those having broad support may instead be described as 'consensual'. As will be observed in the case study in section 9, consensual candidates fare better in a Borda count election than in a plurality rule one whereas the reverse is true for polarized ones. The choice of the PV vector for any given election is critical in influencing whether such candidates are promoted or demoted by this choice. Importantly, can a balanced vector be obtained whereby neither strong support for one candidate nor broad support for another is inherently dominant?

Consider the case where a consensus candidate, C_C, has broader but weaker support than an opposing polarized candidate, C_P. C_P is awarded a preference weighted w_i by a proportion p of all voters. In contrast, C_C receives a lower preference of weight w_{i+k} from a higher proportion q of voters. Does the tally contribution of pw_i for C_P outweigh that of qw_{i+k} for C_C? When they are tied, $pw_i = qw_{i+k}$, hence $p/q = w_{i+k}/w_i$. Given the specific support for each candidate, it is the ratio of the two weightings that determines which has the larger contribution. As the disparity between the two weightings widens, the prospects for C_P improve, while those for C_C advance as this margin narrows.

The bias of a PV vector towards consensus or polarization is not just dependent upon the weighting of one preference relative to another, but is a function of the ratios between all weightings. For any PV vector, the following analysis develops indices that indicate in which direction and to what extent the system is biased. Using them, a vector with a desired bias can be designed. In particular, the use of weightings forming a geometric progression emerges from this analysis. Such vectors can smoothly interpolate between the plurality and Borda count bias extremes regardless of N with just a single variable, the common ratio, as the case study in section 7 demonstrates. The intermediate vector within this spectrum that employs consecutively halved weightings optimizes the balance between strong versus broad candidate support. In section 9, our real-world case study compares this balanced vector against Dowdall's method of using a harmonic sequence of weightings and several advantages emerge.

2. Bias Indices

The relative worth of one rank position compared to another greatly influences the bias of a PV system. Consider two adjacent positions in a normalized vector w where the higher preference has a weight of w_i and the lower preference a weight of w_{i+1}. Two extreme examples and an intermediate one of the disparities between these two weightings are shown in figure 1.

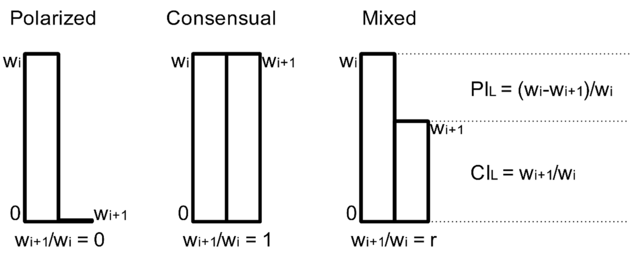

Figure 1. *Examples of polarization and consensus biases.*

In the polarized example, the weighting of the lower-ranked preference is zero so the ratio of the two weightings w_{i+1}/w_i is also zero. The consensus example represents the other extreme. Here the weightings are of the same value so the ratio w_{i+1}/w_i equals one. Therefore, the ratio $r = w_{i+1}/w_i$ varies within the range $0 \leq r \leq 1$. As this ratio increases from zero to one, the differential preference bias changes from being wholly polarized to wholly consensual. Hence, this ratio may be employed as the consensus bias index CI_L for the lower preference. Conversely, as the ratio r decreases, the bias becomes increasingly more polarized. The complementary ratio $1 - r = (w_i - w_{i+1})/w_i$ is thus appropriate as the polarization bias index PI_L for the lower preference. As desired, these two opposing indices always sum to unity as illustrated by the mixed example in figure 1. That is, $CI_L + PI_L = 1$.

Only one of these two indices is strictly required as the other is a dependent variable. However, in some later contexts one index is the more efficient while in others it is the complementary version. This particularly applies to the vector indices that are developed below. Therefore, reference to either or both indices for preferences or for vectors may be cited hereafter.

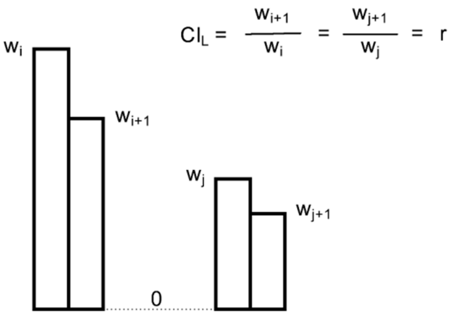

Figure 2. *Differing pairs with common indices.*

The ratio r only refers to the lower preference relative to the higher one in each adjacent pair. There is no reference to the absolute value of either weighting. In figure 2, both pairs of adjacent preferences have the same weightings ratio r but different absolute values. All the preference bias indices affect the overall system bias so all are required to be incorporated into indices for the vector, but a higher-ranked pair needs to contribute more than a lower-ranked one as their weightings are worth more.

The overall vector indices are defined by taking the weighted average of all the individual preference indices. Since the ratio r is referenced with respect to the higher preference, then the weighting of this higher rank is appropriate for this derivation. Each preference pair then contributes $w_i CI_{i+1}$ towards the calculation of the vector consensus index, CI, and $w_i PI_{i+1}$ towards the vector polarization index, PI. These two individual sums are then divided by the sum of all the higher preference weightings to produce the required weighted average.

Let $\Sigma = w_1 + \cdots + w_N$, the sum of the components of a normalized vector w. Recall that $w_1 = 1$ and $w_N = 0$. Then:

$$CI = \frac{\sum_{i=1}^{N-1} w_i CI_{i+1}}{\sum_{i=1}^{N-1} w_i}$$

$$CI = \frac{\sum_{i=1}^{N-1} w_i \left(\frac{w_{i+1}}{w_i}\right)}{\sum_{i=1}^{N-1} w_i}$$

$$CI = \frac{\sum_{i=1}^{N-1} w_{i+1}}{\sum_{i=1}^{N-1} w_i}$$

$$CI = \frac{\Sigma - w_1}{\Sigma - w_N}$$

$$CI = \frac{\Sigma - 1}{\Sigma}$$

Similarly:

$$PI = \frac{\sum_{i=1}^{N-1} w_i PI_{i+1}}{\sum_{i=1}^{N-1} w_i}$$

$$PI = \frac{\sum_{i=1}^{N-1} w_i \left(\frac{w_i - w_{i+1}}{w_i}\right)}{\sum_{i=1}^{N-1} w_i}$$

$$PI = \frac{\sum_{i=1}^{N-1} (w_i - w_{i+1})}{\sum_{i=1}^{N-1} w_i}$$

$$PI = \frac{w_1 - w_N}{\Sigma - w_N}$$

$$PI = \frac{1}{\Sigma}$$

In summary, $CI = 1 - 1/\Sigma$, $PI = 1/\Sigma$ and $CI + PI = 1$. The sum Σ ranges within $1 \leq \Sigma \leq N - 1$ since $1 = w_1 \geq \cdots \geq w_N = 0$. Therefore, the vector indices vary within the range $0 \leq CI \leq (N-2)/(N-1)$ for the consensus index and within $1/(N-1) \leq PI \leq 1$ for the polarization index.

For the plurality vector $(1,0,\ldots,0)$, $\Sigma = 1$; consequently, $PI = 1$ and $CI = 0$. As it is uniquely the most polarized vector, these extreme values accurately reflect this status. The N weightings of the Borda count vector $(1, 1-d, 1-2d, \ldots, 2d, d, 0)$ form an arithmetic progression whose sum Σ is $(w_1 + w_N)N/2 = N/2$. As N increases without bound, the indices vary within the ranges $0 < PI \leq 1$ and $0 \leq CI < 1$. For the anti-plurality vector $(1,\ldots,1,0)$, $\Sigma = N - 1$ and for $N > 2$ it is more consensual than the Borda count since its sum Σ is larger. However, the full range of its bias indices are identical to those for the Borda count as N increases without bound. For the alternative anti-vector version of anti-plurality see section 3.

Another PV method is the unique Dowdall system established in Nauru, whose weightings form a harmonic progression $(1, 1/2, \ldots, 1/(N-1), 1/N)$ [2]. The harmonic sum is defined by $H_N = \sum_{i=1}^{N}(1/i)$. When the Dowdall vector is normalized, its bias index $PI = (1 - 1/N)/(H_N - 1)$. Hence, the index ranges are again $0 < PI \leq 1$ and $0 \leq CI < 1$ as N varies.

With just two candidates, the plurality, Dowdall, Borda count and anti-plurality vectors all normalize to $(1,0)$ so necessarily have the same bias indices; namely $PI = 1/\Sigma = 1$, and $CI = 0$. All PV vectors are therefore equivalent to plurality when there is a straight fight between two competitors. As N increases, the CI for all these vectors also increases towards unity; except for plurality which remains wholly polarized. Hence, each becomes ever more consensual as the field of candidates expands. The ultimate consensus vector, the indifference vector $(1,\ldots,1)$, is invalid since its first and last preferences are equal. Indeed, all its preferences are identical so it is wholly consensual and thus cannot distinguish between any of the candidates. It is therefore fortunate that the Dowdall, Borda count and anti-plurality vectors are never quite fully consensual.

3. Anti-Vectors and Conjugate Vectors

To every normalized vector $w = (w_1, w_2, \ldots, w_{N-1}, w_N)$

we define a corresponding anti-vector w^A and conjugate vector w^C by:

$$\mathrm{w}^A = (-w_N, -w_{N-1}, \dots, -w_2, -w_1)$$

$$\mathrm{w}^C = (1 - w_N, 1 - w_{N-1}, \dots, 1 - w_2, 1 - w_1)$$

Note: 1) both these operations are involutory: namely, $(\mathrm{w}^A)^A = \mathrm{w}$ and $(\mathrm{w}^C)^C = \mathrm{w}$; 2) the anti-vector and conjugate vector differ by the indifference vector, hence are affine equivalent; 3) the conjugate vector is normalized.

Anti-plurality is the complementary voting system to plurality in that each voter casts a favorable vote for just one candidate in a plurality election, but casts an unfavorable vote against just one candidate in an anti-plurality election [1]. When using PV vectors, voters are casting positive preferences in favor of candidates rather than negative ones against them. For PV anti-vectors, it is the other way round.

As the plurality (PL) vector is $(1,0,\dots,0)$, its anti-vector, A-PL, is $(0,\dots,0,-1)$, and its conjugate vector is $(1,\dots,1,0)$. For anti-plurality, either of these affine equivalent versions may be used in practice. As the Borda count (BC) vector is $(1,1-d,\dots,d,0)$, its anti-vector, A-BC, is $(0,-d,\dots,d-1,-1)$ and its conjugate vector is $(1,1-d,\dots,d,0)$. Note that the BC vector is self-conjugate. As any anti-vector and its corresponding conjugate vector are affine equivalent, their bias indices are identical.

Since PL and BC are opposed extremes in terms of vector bias, their anti-vectors and conjugate vectors are similarly opposed extremes. The Borda count's self-conjugacy connects both ranges at this common extreme. So, this full spectrum is a function of Σ if anti-vectors are normalized into their equivalent vector format, the conjugate vector. By adding one to each non-positive weighting to achieve this conversion, N is thereby added to the sum of the weightings. For A-BC, this anti-vector has a sum of $-N/2$, while the conjugate vector has a sum $\Sigma = N - N/2$. Hence the BC vector, and necessarily every self-conjugate one, must have the same sum $\Sigma = N/2$. Since Σ varies within the range $1 \le \Sigma \le N-1$, all possible normalized PV vectors are included within the bias spectrum from the wholly-polarized PL vector, via the BC vector and A-BC anti-vector, to the maximally-consensual A-PL anti-vector for any given number of candidates.

4. Designing a Positional Voting System for a Specific Bias

Having defined the bias indices for any PV system, it is then possible not only to compute the bias associated with a particular vector or anti-vector but also to design one that yields the desired system bias. By first specifying the required polarization index subject to $0 < PI \le 1$, the sum Σ is determined as $1/PI$. Once it has been identified, any valid set of weightings yielding this total will have the requisite bias. However, many vectors generating different overall candidate rankings will have an identical system bias. For example, the Vote-for-3-out-of-6 vector $(1,1,1,0,0,0)$ produces the same indices as that of a six-candidate BC, since for both, the sum $\Sigma = 3 = N/2$. The selection of the individual weightings comprising Σ will characterize not just its bias but other features of the system such as susceptibility to cloning; see section 10.

For a rigorous approach to deriving weightings with a chosen bias, reference should be made to the original derivation of the indices in section 2. Each vector index (CI or PI) is the weighted average of all the individual indices (CI_L or PI_L) associated with the vector. By choosing the same value for every individual index, the weighted average of them for the vector index necessarily has the same value.

Using this systematic PV design approach, each $PI_L = PI = 1/\Sigma$ and each $CI_L = CI = 1 - 1/\Sigma = (\Sigma - 1)/\Sigma$. Since by definition $CI_{i+1} = w_{i+1}/w_i$, then this common ratio $w_{i+1}/w_i = r = (\Sigma - 1)/\Sigma$. A sequence of weightings with a common ratio r between adjacent preferences forms a standard geometric progression. Such a vector v has the format $(1, r, \dots, r^{N-2}, r^{N-1})$ where the weighting of the first preference is unity. Solving $r = (\Sigma - 1)/\Sigma$ for Σ yields $\Sigma = 1/(1-r)$. Thus, $r = 1$ is not permitted, but it may vary within the range $0 \le r < 1$ to produce valid vectors. Note that if $r = 1$ was used, the vector would be affine equivalent to the invalid indifference one. Also, $r > 1$ is impermissible since lower preferences would then have greater weight than higher ones. Since it would result in an invalid sequence of alternating positive and negative weightings, $r < 0$ is similarly not permitted.

5. Geometric Voting

Let a valid vector with a common ratio r be referred to as a GV(r) vector, where GV stands for Geometric Voting. The standard GV $(0 \le r < 1)$ vector $\mathrm{v} = (1, r, \dots, r^{N-2}, r^{N-1})$. Consider the case of an election with infinitely many candidates. The sum to infinity of the vector weightings is the sum of the geometric series, $S_\infty = 1/(1-r)$. From this sum, the two bias indices are:

$$PI = 1/\Sigma = 1/S_\infty = 1 - r$$

$$CI = 1 - PI = r$$

For a GV(r) vector, its consensus index approaches its common ratio as N tends to infinity. For an election with N candidates, the sum of the vector weightings is defined by $S_N = (1 - r^N)/(1 - r)$. Since $0 \le r < 1$, as N increases, r^N decreases towards zero, so S_N approaches $1/(1-r)$. Therefore, designing a PV system with a bias of $CI = r$ only requires a GV(r) vector to be adopted, provided a large enough field of candidates is anticipated.

Recall that the standard GV(r) vector v has not been normalized, since the value of its lowest preference v_N is not zero but r^{N-1}. When it is close to zero, there is a negligible reduction in the value of each weighting following normalization. Strictly, however, the sum on which the bias indices are based is the sum Σ of all the normalized vector weightings. Hence, when v_N is not close to zero, these indices must be accurately calculated. The derivation of the sum Σ for a normalized GV(r) vector w is shown below.

As the standard GV(r) vector v is $(1, r, \dots, r^{N-2}, r^{N-1})$, the normalized GV($r$) vector w is:

$$\frac{1}{1-r^{N-1}}(1-r^{N-1}, r-r^{N-1}, \ldots, r^{N-2}-r^{N-1}, 0)$$

Since Σ is the sum of all the normalized preference weightings, then:

$$\Sigma = \frac{1}{1-r^{N-1}}\left(\sum_{i=0}^{N-1} r^i - Nr^{N-1}\right)$$

$$\Sigma = \frac{1}{1-r^{N-1}}\left(\frac{1-r^N}{1-r} - Nr^{N-1}\right)$$

$$\Sigma = \frac{(1-r^N)-(1-r)Nr^{N-1}}{(1-r)(1-r^{N-1})}$$

$$\Sigma = \frac{1}{1-r}\left(\frac{1-r^N-Nr^{N-1}+Nr^N}{1-r^{N-1}}\right)$$

$$\Sigma = \frac{1}{1-r}\left(\frac{1-r^N-r^{N-1}-(N-1)r^{N-1}+Nr^N}{1-r^{N-1}}\right)$$

$$\Sigma = \frac{1}{1-r}\left(1-\frac{(N-1)r^{N-1}-(N-1)r^N}{1-r^{N-1}}\right)$$

$$\Sigma = \frac{1}{1-r}\left(1-\frac{(1-r)(N-1)r^{N-1}}{1-r^{N-1}}\right)$$

$$\Sigma = \frac{1}{1-r} - \frac{(N-1)r^{N-1}}{1-r^{N-1}}$$

Thus, for fixed r between 0 and 1, as N increases, Σ approaches $1/(1-r)$.

Figure 3 illustrates how the consensus bias index of a GV(r) vector varies according to the number of candidates, for a selection of common ratios from its minimum value, $r=0$, through to its maximum, $r=1-\delta$, where $\delta \to 0$.

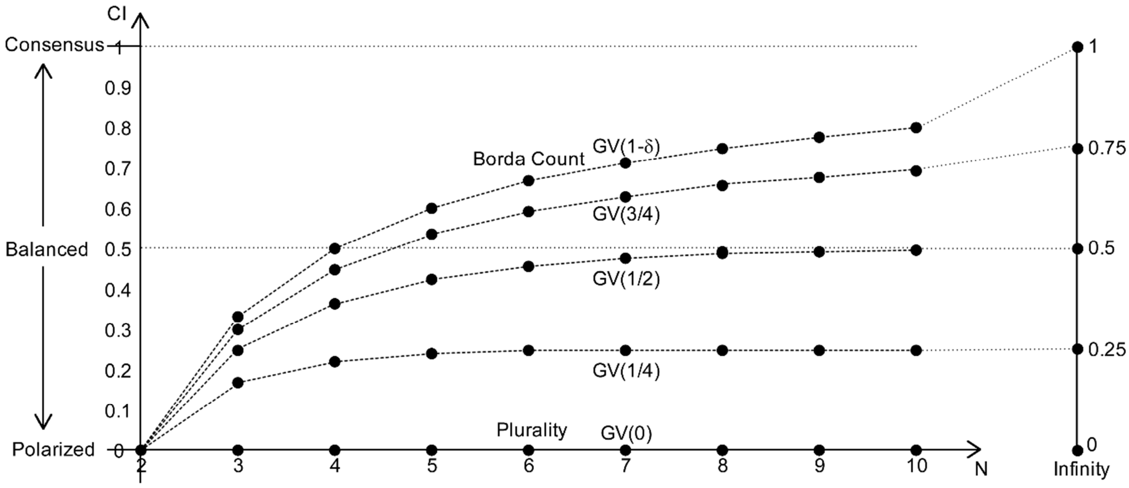

Figure 3. *Examples of GV(r) vector bias.*

For the standard GV(r) vector v, there is always the common ratio r between adjacent preferences. However, the process of normalization disturbs these individual ratios such that they are no longer identical. Indeed, the consensus bias index for the lowest preference P_N is always $CI_N = 0$ and not r. These disturbances increase as N decreases or as r increases. The gap between the actual value of the consensus index for a GV(r) vector and that of its limit, $CI = r$, is visible in figure 3. For low common ratios and a sufficiently large field of candidates, this gap is insignificant.

The GV(0) vector $(1,0,\ldots,0)$ is identical to the PL vector. At the other extreme, the common ratio approaches one; namely, $r = 1-\delta$, where $\delta \to 0$. Here, the standard GV(r) vector is $(1, 1-\delta, (1-\delta)^2, \ldots, (1-\delta)^{N-1})$. By the binomial theorem, the $(k+1)$-st component of the vector is $1-k\delta+O(\delta^2)$. As $\delta \to 0$, for fixed N, terms of the form $O(\delta^2)$ may be ignored. Therefore, as $r \to 1$, for fixed N, the GV(r) vector is approximated by the arithmetic progression $(1, 1-\delta, 1-2\delta, \ldots, 1-(N-1)\delta)$. This resultant vector is therefore identical to a BC vector where δ is the common difference, and where $\Sigma \to N/2$ when the weightings are normalized. Hereafter, the notation GV(→1) is used to denote GV(r) with $r \to 1$.

Standard GV $(0 \le r < 1)$ vectors have normalized weightings sums in the range $1 \le \Sigma < N/2$. At the lower end of both ranges, the GV(0) vector with $\Sigma = 1$ is the PL vector. At the higher end of both ranges, the GV(→1) vector with $\Sigma \to N/2$ approaches a BC vector. Thus, GV(r) vectors where $1 \le \Sigma < N/2$ provide a one-parameter family of PV vectors smoothly interpolating between PL and BC for any given number of candidates.

The anti-vector of the standard GV(r) vector $(1, r, \ldots, r^{N-2}, r^{N-1})$ is $(-r^{N-1}, -r^{N-2}, \ldots, -r, -1)$, and its conjugate vector is $(1-r^{N-1}, 1-r^{N-2}, \ldots, 1-r, 0)$. The anti-vector of the GV(0) vector $(1,0,\ldots,0)$, A-GV(0), is $(0,\ldots,0,-1)$. Hence, A-GV(0) is equivalent to A-PL. Similarly, for the normalized GV(→1) vector $(1, 1-d, \ldots, d, 0)$, its anti-vector A-GV(→1) is $(0, -d, \ldots, d-1, -1)$. Therefore, A-GV(→1) is equivalent to A-BC. Thus, A-GV(r) anti-vectors provide a one-parameter family of PV anti-vectors smoothly interpolating between A-BC and A-PL. As these anti-vectors are affine equivalent to conjugate vectors where $N/2 < \Sigma \le N-1$, then, for $1 \le \Sigma \le N-1$, the whole bias spectrum from PL via BC/A-BC to A-PL is essentially encompassed for any given number of candidates.

As an example, figure 4 shows the weightings for the standard ten-candidate GV(1/2) vector v, its anti-vector A-GV(1/2), and its conjugate vector. The consensus index for this vector v is approximately 0.495558, very close to its asymptotic value of $CI = 0.5$. This mid-range value for both the common ratio and the bias indices, $r = 1/2 = CI = PI$, produces a balanced vector where the bias towards polarization is cancelled out by that towards consensus.

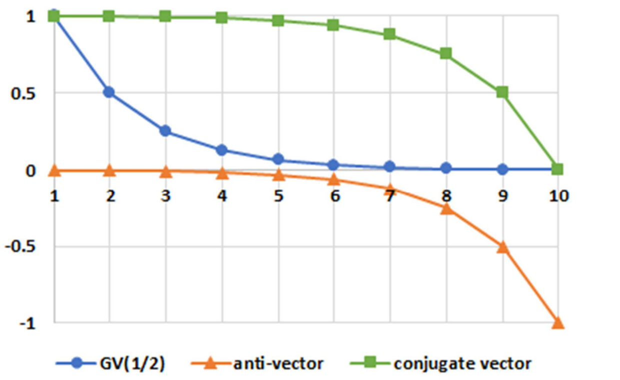

Figure 4. *Weightings of GV(1/2), its anti-vector, and conjugate vector, for* $N = 10$.

As all GV(r) vectors and A-GV(r) anti-vectors constitute a subset of PV ones, they satisfy all the voting system criteria and match all the other characteristics of positional voting that these PV vectors collectively do. However, only one PV vector satisfies the reversal symmetry criterion; namely, the Borda count [1]. This criterion requires that the ranking of the candidate tallies be reversed when the voters' ranking preferences are reversed. This property is not valued in practice, however. For single-winner elections, it is far more consequential whether candidate C finishes in first or second place, compared to whether C is in last or penultimate place after reversal [3].

6. Equivalent Three-Preference Positional Voting Systems

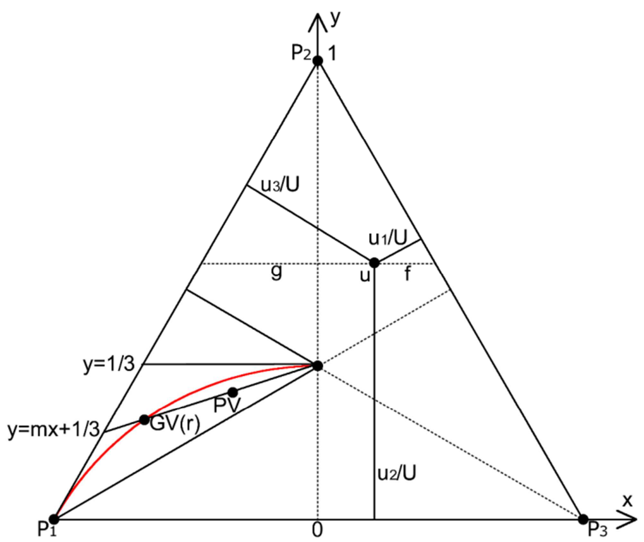

Figure 5. *Three-preference vector map.*

For PV elections with just three candidates standing, the equilateral-triangular map shown in figure 5 can represent any possible vector u, with the three preferences $P_1 = (1,0,0)$, $P_2 = (0,1,0)$, and $P_3 = (0,0,1)$; whether valid or not. A barycentric coordinate system is used for measuring perpendicular distances from the respective sides of the triangle, with coordinates summing to one. A general vector projects onto this plane using the rule $\mathrm{u} = (u_1, u_2, u_3)/U$, where $U = u_1 + u_2 + u_3$. The GV(r) vector u is represented by $(1, r, r^2)/(1 + r + r^2)$.

A rectangular coordinate system is superimposed onto figure 5. The x-axis is the directed line through P_1 and P_3. The y-axis perpendicularly bisects the edge connecting P_1 and P_3, and is directed towards P_2. The origin is the midpoint of the edge connecting P_1 and P_3.

Next, g is defined as the directed horizontal distance measured leftward from u to the edge connecting P_1 and P_2, and f is the directed horizontal distance measured rightward from u to the edge connecting P_2 and P_3. The cartesian coordinates for any vector u are given below:

$$y = \frac{u_2}{U}$$

$$x = \frac{g-f}{2} = \frac{1}{2}\left(\frac{2u_3}{U\sqrt{3}} - \frac{2u_1}{U\sqrt{3}}\right) = \frac{u_3 - u_1}{U\sqrt{3}}$$

Of the six congruent ranking regions within the map representing the six possible orderings of the three barycentric coordinates, only the one containing the point PV (corresponding to a specific PV vector) represents valid vectors, since only here is $u_1 \geq u_2 \geq u_3$. The map's center where $u_1 = u_2 = u_3$ represents the invalid indifference vector, and is not considered part of any ranking region.

Consider the vectors $(1 - y, y, 0)$ where $0 \leq y \leq 1/2$. On the map, the locus of these vectors is the lower half of the edge connecting P_1 and P_2, where $u_3 = 0$. It is also one of the three boundaries of the valid ranking region. In terms of collective candidate rankings, Saari has shown that any valid PV vector is equivalent to a vector in this half-edge [1]. When plotted on the map in figure 5, all these equivalent vectors are located at points along a line between the map's edge at $(1 - y, y, 0)$ and its center at $(1/3, 1/3, 1/3)$; see the line $y = mx + 1/3$ on the map. The affine equivalence of two vectors is demonstrated by projection, from the triangle's center through its interior onto its boundary. An algebraic proof of this statement follows below.

Consider these two affine equivalent vectors:

$$\frac{(u_1, u_2, u_3)}{U} \equiv \frac{(u_1 + k, u_2 + k, u_3 + k)}{U + 3k}$$

Substituting the cartesian coordinates (x, y) for each vector yields:

$$\left(\frac{u_3 - u_1}{U\sqrt{3}}, \frac{u_2}{U}\right) \equiv \left(\frac{u_3 - u_1}{(U + 3k)\sqrt{3}}, \frac{u_2 + k}{(U + 3k)}\right)$$

The line $y = mx + 1/3$ has slope m, and the triangle's center is $(0, 1/3)$ in rectangular coordinates. Then:

$$m = \frac{y - {}^{1}/_{3}}{x} = \frac{\left(\frac{u_2}{U} - \frac{1}{3}\right)}{\left(\frac{u_3 - u_1}{U\sqrt{3}}\right)} = \frac{\left(\frac{u_2 + k}{U + 3k} - \frac{1}{3}\right)}{\left(\frac{u_3 - u_1}{(U + 3k)\sqrt{3}}\right)}$$

$$\frac{\left(\frac{u_2}{U}-\frac{1}{3}\right)}{\left(\frac{u_3-u_1}{U}\right)} = \frac{\left(\frac{u_2+k}{U+3k}-\frac{1}{3}\right)}{\left(\frac{u_3-u_1}{U+3k}\right)}$$

$$\frac{\left(u_2-\frac{U}{3}\right)}{u_3-u_1} = \frac{\left(u_2+k-\frac{U+3k}{3}\right)}{u_3-u_1}$$

$$u_2-\frac{U}{3} = u_2+k-\frac{U}{3}-k$$

$$u_2-\frac{U}{3} = u_2-\frac{U}{3}$$

Reversing the steps from the final identity, the original equation is established. Therefore, any two vectors along this straight line through the valid ranking region from the map's edge to its center are equivalent as regards collective candidate rankings.

Further, every valid PV vector is located on such a line. As the vertex P_1 represents the PL vector $(1,0,0)$, all points along the line from here to the map center are PL equivalent, as they do not distinguish between second and third place rankings. Similarly, all points along the line from $(1/2,1/2,0)$ to the map center are equivalent to the anti-plurality conjugate vector $(1,1,0)$, as they do not distinguish between first and second place rankings. These two lines are the other boundaries of the valid ranking region. The horizontal straight line $y = 1/3$ represents all the BC-equivalent vectors $(y+d, y, y-d)$ where $0 < d \leq 1/3$.

With non-negative weightings, GV(r) vectors may also be plotted on a three-preference map. Using the GV(r) vector u as given below ensures that the three preference coordinates sum to unity:

$$\mathrm{u} = \frac{(u_1,u_2,u_3)}{u_1+u_2+u_3} = \frac{(1,r,r^2)}{1+r+r^2}$$

The cartesian coordinates (x,y) for vector u are:

$$\left(\frac{u_3-u_1}{U\sqrt{3}}, \frac{u_2}{U}\right) = \left(\frac{r^2-1}{(1+r+r^2)\sqrt{3}}, \frac{r}{(1+r+r^2)}\right)$$

As the common ratio varies from 0 towards 1, what trajectory on the map does the GV(r) vector u follow? As shown by the red curve on the map, the locus forms a circular arc with a radius of $2/3$ centered at $(x,y) = (0,-1/3)$, or $(2/3, -1/3, 2/3)$ in barycentric coordinates. Using the equation for a circle, the following proof confirms this statement:

$$\left(\frac{2}{3}\right)^2 = (x-0)^2 + \left(y+\frac{1}{3}\right)^2$$

$$\frac{4}{9} = \left(\frac{r^2-1}{(1+r+r^2)\sqrt{3}}\right)^2 + \left(\frac{r}{1+r+r^2}+\frac{1}{3}\right)^2$$

$$\frac{4}{9} = \frac{1}{3}\left(\frac{r^2-1}{1+r+r^2}\right)^2 + \left(\frac{3r+1+r+r^2}{3(1+r+r^2)}\right)^2$$

$$4 = 3\left(\frac{r^2-1}{1+r+r^2}\right)^2 + \left(\frac{1+4r+r^2}{1+r+r^2}\right)^2$$

$$4(1+r+r^2)^2 = 3(r^2-1)^2 + (1+4r+r^2)^2$$

$$4\big((1+r+r^2)+(r+r^2+r^3)+(r^2+r^3+r^4)\big) = 3(r^4-2r^2+1)+(1+4r+r^2)+(4r+16r^2+4r^3)+(r^2+4r^3+r^4)$$

$$4+8r+12r^2+8r^3+4r^4 = 4+8r+12r^2+8r^3+4r^4$$

Reversing the steps from the final identity, the original equation is established; thus, the circular arc displayed on the map is indeed the locus of GV$(0 \leq r < 1)$ vectors.

For PV vectors formed as a non-negative linear combination of PL and BC, any positive-slope line in the valid ranking region from the map's edge to its center only intersects the arc of GV(r) vectors at one unique point. Therefore, the GV(r) vector represented by this point is equivalent to all the other PV vectors identified by this line. The common ratio here is derived as follows from the definition of the weightings:

$$\mathrm{u} = \frac{(u_1,u_2,u_3)}{u_1+u_2+u_3} = \frac{(1,r,r^2)}{1+r+r^2}$$

$$\frac{u_1-u_2}{U} = \frac{1-r}{1+r+r^2}$$

$$\frac{u_2-u_3}{U} = \frac{r(1-r)}{1+r+r^2}$$

$$r = \frac{u_2-u_3}{u_1-u_2}$$

Therefore, for any three-preference PV vector that is a non-negative linear combination of PL and BC, there is an equivalent GV(r) vector as defined below:

$$PV(u_1,u_2,u_3) \equiv GV\left(r = \frac{u_2-u_3}{u_1-u_2}\right)$$

Here, the compound inequality $0 \leq r < 1$ can be re-written: $0 \leq (u_2-u_3)/(u_1-u_2) < 1$. Since the numerator and denominator of the fraction are non-negative, it follows that $u_2-u_3 < u_1-u_2$. On the triangle's edge where $u_3 = 0$, this simplifies to $u_2 < u_1-u_2$, or $2u_2 < u_1$. Since the sum of the coordinates is one, it follows that the projection of the GV$(0 \leq r < 1)$ arc from the triangle's center onto its edge has range equal to the line segment from (1,0,0) to $(2/3, 1/3, 0)$ in barycentric coordinates.

As mentioned earlier, Saari has similarly demonstrated that the vector $(1-s, s, 0)$ is affine equivalent to (u_1,u_2,u_3) where $s = (u_2-u_3)/(u_1+u_2-2u_3)$ [1]. With only three preferences, just a single variable (s or r) is sufficient to allow an equivalent vector to represent one anywhere between PL and BC. However, for more preferences, $N-2$ variables are needed. Nevertheless, with just the sole variable r, GV(r) vectors can represent some - but not all – vectors on a continuous system bias spectrum smoothly interpolating between PL and BC for any number of preferences. In section 7, an example of such a spectrum is provided for an historic election in which four candidates competed.

7. Case Study: 1860 US Presidential Election

The United States presidential elections of 1824 and 1860 were the only ones in which four presidential candidates earned electoral votes (as opposed to spurious ones cast by faithless electors). The 1860 election is particularly important for its connection to the subsequent American Civil War. It has been studied by historians in order to understand the influence of the choice of voting system. Different ones might have led to a different election outcome, possibly avoiding war, but also possibly prolonging slavery.

The result of a PV election could only be calculated with a hypothetical profile for the voters of 1860. Such a profile would need to be consistent with existing information about vote totals, which enumerate the first choices of voters. Saari describes two attempts that were made by historians to estimate the complete voter profile: the second, third, and fourth choices of voters whose first choices were known [4]. One such profile was constructed by Riker; the other by a team of historians [5].

The four candidates were Abraham Lincoln (Republican), Stephen Douglas (Democratic), John Breckinridge (Southern Democratic), and John Bell (Constitutional). The profiles by Riker and by the team of historians are largely similar, except for a dispute regarding the ranking by Lincoln voters. Both teams agree that they would place Breckenridge in last place. However, Riker believed that most Lincoln voters preferred Bell over Douglas; in contrast, the second team believed that they were about evenly split between Bell and Douglas as a second choice.

For simplicity, we studied the mean of the two profiles. Our study is based on an imaginary contest in the popular vote, as opposed to the constitutional procedure of an electoral college vote.

Figure 6 depicts the effect of changing r on the four components of the normalized GV(r) vector w.

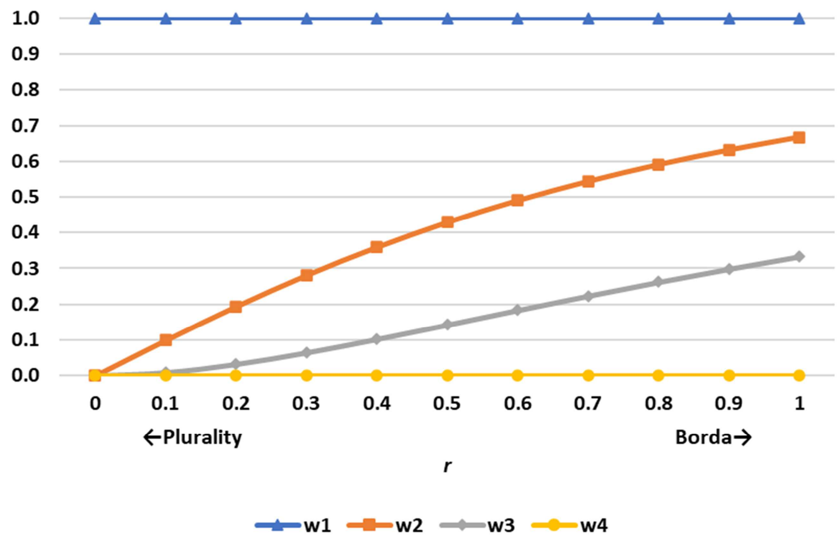

Figure 6. *Components of normalized GV(r) vector w for N = 4.*

In describing vote share, it is useful to see the same components renormalized such that their sum is equal to one; as in the previous section. Figure 7 displays the components for the resultant GV(r) vector $u = (u_1, u_2, u_3, u_4) = (w_1, w_2, w_3, w_4)/(w_1 + w_2 + w_3 + w_4)$.

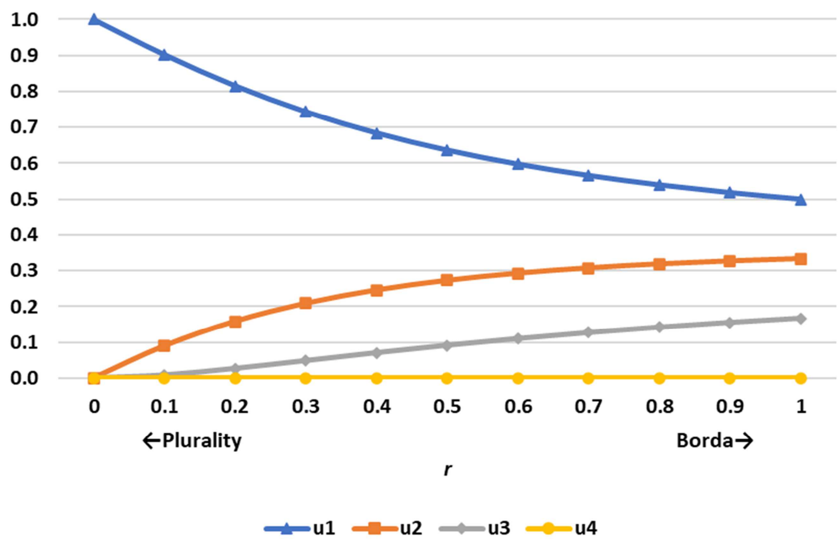

Figure 7. *Components of resultant GV(r) vector u.*

Finally, the components of the vector u shown in figure 7 are applied to the hypothetical profile described above, and the percentage vote share computed for each candidate under GV(r) for all r between 0 and 1; see figure 8.

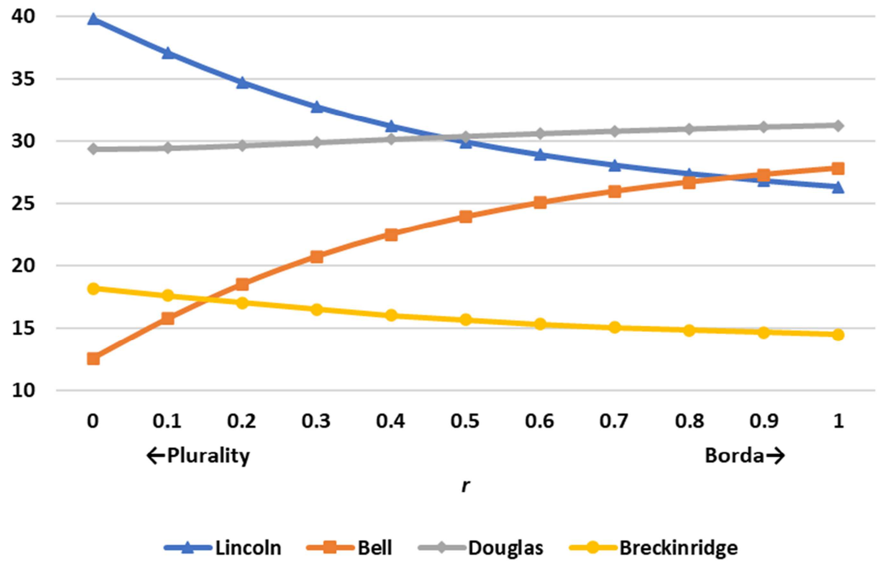

Figure 8. *Popular vote in the 1860 US presidential election under GV(r).*

The plurality-winner Lincoln appears as a strong but polarizing candidate, while Breckinridge seems to have had little support outside the South, nor could he defeat Lincoln at any value of r. Similarly, Bell could not defeat Douglas at any value of r. The consensus candidate Douglas however maintains a consistently good performance at all values of r and would have won a Borda count contest. Indeed, his popular vote share surpasses that for Lincoln using a common ratio of 1/2 or higher.

8. Consecutively Halved Positional Voting

Let the GV(1/2) vector be called the Consecutively Halved Positional Voting or CHPV vector as it is an important and special case of GV. For a balanced PV system, neither broad support for one candidate nor strong support for a competitor is allowed to dominate the other. Using CHPV, a weighting at any rank position is always worth the same as two at the adjacent lower rank; namely, $w_i = 2w_{i+1}$. Whether a candidate campaigns to upgrade a preference to the next higher rank or to gain an extra preference at the same rank instead, the additional tally contribution is identical. Hence, stronger, or broader support here is of equal value.

For a precisely balanced vector, $CI = PI = 1/2$. Hence, $\Sigma = 1/PI = 2$. Also, $\Sigma = w_1 + \sum_{i=2}^{N} w_i$. Since $\Sigma = 2$ and $w_1 = 1$, then $\sum_{i=2}^{N} w_i = 1$. For a normalized GV(r) vector w, $\sum_{i=2}^{N} w_i = w_1$. In other words, the first preference must be worth the same as all the remaining ones in total to achieve balance. CHPV is the sole PV vector v to have the unique property that any one of its standard weightings is always equal to the sum to infinity of all the lower weightings; that is, $v_i = \sum_{n=1}^{\infty} v_{i+n}$ for all i such that $1 \leq i \leq N$. For the first preference for example, $1 = 1/2 + 1/4 + 1/8 + \cdots + 1/\infty$. Therefore, since $v_i \to w_i$ as $v_N \to w_N = 0$, the only solution to $w_1 = \sum_{i=2}^{\infty} w_i = \sum_{i=2}^{\infty} v_i = \sum_{i=2}^{\infty} r^{i-1} = 1$ is where $r = 1/2$. This sum exceeds one when $r > 1/2$ and is less than one when $r < 1/2$. For a practical election with a sufficiently large number of candidates, $\sum_{i=2}^{N}(1/2)^{i-1} \cong 1$ and the approximation asymptotically converges on unity as N increases; see the disparity between $CI = 1/2$ and the CI curve for GV(1/2) in figure 3. CHPV becomes ever more balanced as the field of candidates expands and is the only GV(r) vector to converge towards exact balance.

Consider an election employing the standard CHPV vector v where a polarized candidate, C_P, is awarded first preference ($v_1 = 1$) by a proportion p of voters, and last preference ($v_N \to 0$) by the remaining voters. Amongst other candidates, a consensus candidate, C_C, gains all the remaining first preferences and additionally second preferences ($v_2 = 1/2$) from the other voters; its strongest possible challenge to C_P. The proportional tally for C_C is then $(1-p)1 + p(1/2) = 1 - p/2$, and for C_P it is at least p. These tallies are equal when $p = 2/3$. Therefore, with $v_N \to 0$, the polarized candidate requires at least a two-thirds majority of first preferences to ensure victory regardless of how many candidates compete or how other votes are cast; otherwise, the consensus candidate wins.

Plurality satisfies the majority consistency criterion but the Borda count does not [6]. To pass this criterion, any candidate with a tally higher than the total received by all the other candidates must win. Using the polarized PL vector, a candidate only needs to exceed a simple majority to win; so, satisfying this criterion. Whereas for the BC vector, a consensus candidate may still win even when the vast majority of first preferences are awarded to a polarized candidate [7]. The candidate with the most first preferences has the strongest support and wins a PL election while the candidate with the highest average rank position has the broadest support and wins a BC election. As CHPV satisfies the two-thirds majority criterion instead, any candidate must exceed this higher threshold for victory to be guaranteed. The winner in CHPV is determined by the balance of strong versus broad support for each candidate.

Figure 9 displays the consensus bias indices for the five major voting systems addressed herein: plurality, CHPV, Dowdall, the Borda count, and anti-plurality. For PL ≡ GV(0), $CI = 0$, its minimum value, whereas anti-plurality has the maximum valid value $CI = (N-2)/(N-1)$. For BC ≡ GV(→1), we compute $CI = (N-2)/N$; the graph demonstrates that there is little difference between BC and anti-plurality for large N as measured by the consensus bias index. Note that for the anti-plurality and Borda count vectors with $N = 3$ and $N = 4$ respectively both are balanced but, unlike CHPV, they both become markedly unbalanced when one or more (clone) candidates are also nominated; see section 10. The bias index for CHPV rises rapidly towards its asymptotic value of $1/2$ as N increases so becoming ever more balanced. In contrast, the Dowdall system does not establish a clear tendency towards either polarization or consensus. Its CI rises slowly, approaching one as N increases. For $N \le 8$, Dowdall is more polarized than consensual; otherwise, the reverse is true.

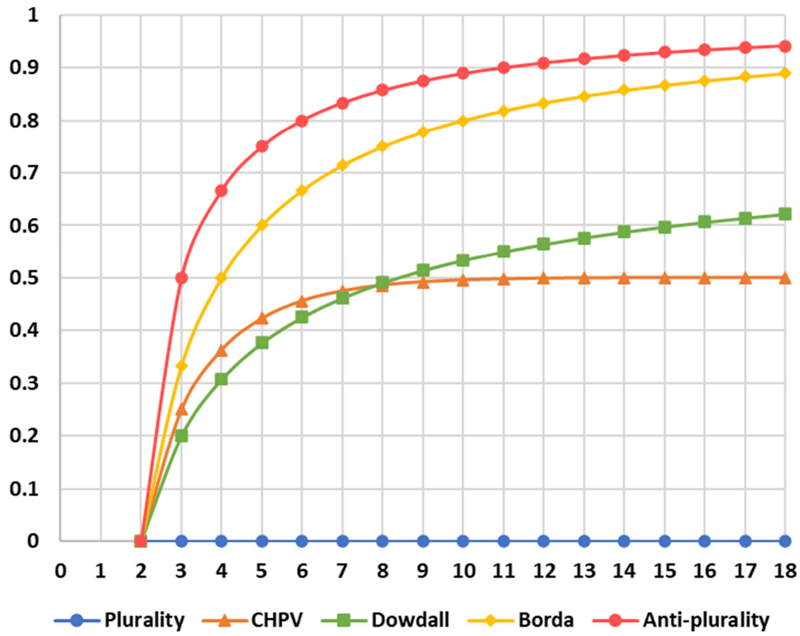

Figure 9. *Consensus bias indices (CI) versus N for major voting systems.*

Several forms of the CHPV vector are tabulated for small values of N, together with related data for the normalized vectors, in table 1.

Table 1. *CHPV vectors.*

Number of candidates	Standard CHPV vector	Normalized CHPV vector	Σ	*PI*	*CI*
2	$\left(1,\frac{1}{2}\right)$	(1,0)	1	1	0
3	$\left(1,\frac{1}{2},\frac{1}{4}\right)$	$\left(1,\frac{1}{3},0\right)$	$\frac{4}{3}$	$\frac{3}{4}$	$\frac{1}{4}$
4	$\left(1,\frac{1}{2},\frac{1}{4},\frac{1}{8}\right)$	$\left(1,\frac{3}{7},\frac{1}{7},0\right)$	$\frac{11}{7}$	$\frac{7}{11}$	$\frac{4}{11}$
5	$\left(1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16}\right)$	$\left(1,\frac{7}{15},\frac{3}{15},\frac{1}{15},0\right)$	$\frac{26}{15}$	$\frac{15}{26}$	$\frac{11}{26}$
N	*i*-th component: $\frac{1}{2^{i-1}}$	*i*-th component: $\frac{2^{N-i}-1}{2^{N-1}-1}$	$\frac{2^N-N-1}{2^{N-1}-1}$	$\frac{2^{N-1}-1}{2^N-N-1}$	$\frac{2^{N-1}-N}{2^N-N-1}$

The numerator of Σ (and thus the denominator of both PI and CI) in the last row represents the values of a sequence that appears three times in the Online Encyclopedia of Integer Sequences [8].

The general vectors for PL and BC, and both the anti-vector and conjugate vector versions for anti-plurality, all use integer weightings in practical elections to assist counting but the Dowdall vector $v = (1, 1/2, 1/3, \dots)$ does not. Fractions such as 1/3 and 1/7 require rounding to a specified number of decimal places prior to counting. As full accuracy is not practical, errors in candidate ties or rankings in tight contests cannot be ruled out however remote the possibility.

Although the standard CHPV vector $v = (1, 1/2, 1/4, \dots)$ also employs fractions, rounding is never required or permitted. Where decimal numbers are used in a manual count or binary numbers in an electronic one, the number of places after the decimal or binary point needed to represent v_i is $i-1$; see table 2. Alternatively, to avoid fractions altogether, the affine equivalent vector $(\dots, 8, 4, 2, 1)$ to v may be employed instead. Therefore, full accuracy can always be maintained with CHPV and only genuine candidate ties need ever occur.

Table 2. *Decimal and binary values for CHPV fractional weightings.*

i	CHPV weighting v_i		
	Fraction	Decimal	Binary
1	1	1	1
2	1/2	0.5	0.1
3	1/4	0.25	0.01
4	1/8	0.125	0.001
5	1/16	0.0625	0.0001
6	1/32	0.03125	0.00001

9. Case Study: Nauru 2019 Parliamentary Elections

On August 24, 2019 a general election was held in the Republic of Nauru across all eight multiple-winner constituencies to fill the 19 seats in their Parliament. Voting is compulsory and each voter must cast a complete preferential ranked ballot without truncating or duplicating any preferences. The Dowdall method of determining the outcome was again employed and the results of this general election were published in a report by the Nauru Electoral Commission [9].

In order to compare the PL, CHPV, BC and Dowdall vectors in a real-world scenario, we reran the eight elections with the same voter input using the three alternative rules. Given the number of candidates that stood in the 2019 elections, table 3 lists the consensus bias index for each of the eight Dowdall contests. For comparison, the corresponding indices for the CHPV and BC rules are also included. The index for PL is always $CI = 0$.

Table 3. *Comparative consensus bias indices.*

Constituency	*N*	Consensus Index (*CI*)		
		Dowdall	CHPV	BC
Aiwo	8	0.491	0.487	0.750
Anabar	6	0.425	0.456	0.666
Anetan	8	0.491	0.487	0.750
Boe	5	0.377	0.423	0.600
Buada	7	0.462	0.475	0.714
Meneng	9	0.514	0.492	0.777
Ubenide	12	0.564	0.499	0.833
Yaren	5	0.377	0.423	0.600

The bias indices for a given N-candidate PV election indicate whether polarized, balanced or consensus candidates may potentially be promoted or demoted by the choice of election rule. For PL with $CI = 0$, polarized candidates are consistently advantaged. The BC consensus index here ranges from 0.6 to 0.833 so benefitting consensus candidates. As an essentially balanced system, the CI for CHPV ranges narrowly between 0.423 and 0.499; close to its ideal 0.5 asymptotic value. The CI for Dowdall includes this range but extends further at both ends; namely, from 0.377 to 0.564.

The effect that vector choice has on candidates' prospects is witnessed in the nine-candidate election in the three-seat Meneng constituency; see table 4. In the Dowdall row, the nine candidates are labelled and listed in the collective rank order that resulted from the 2019 election. Using CHPV instead, the resultant ranking, including the three winners, is unchanged. Note that both these rules have a bias close to the balanced value of $CI = 0.5$ and that the polarized PL and consensual BC rules produce significantly different rankings and winners. For example, the official second-placed candidate (2) and joint winner would have come fourth and lost under PL but top under BC.

Table 4. *Comparative candidate rankings for Nauru 2019 Meneng election.*

	Winners			Losers						*CI*
PL	1	3	4	2	6	5	9	8	7	0
CHPV	1	2	3	4	5	6	7	8	9	0.492
Dowdall	1	2	3	4	5	6	7	8	9	0.514
BC	2	4	5	7	1	3	6	8	9	0.777
	1	2	3	4	5	6	7	8	9	
	Rank position (*i*)									

In figure 10, the preference tally for each of the nine preferences is plotted for the three top-ranked candidates (1, 2, and 3) in the Dowdall election. Candidates 1 and 3 may be described as more polarized than consensual as they each attracted both a large share of first preferences, a large share of last preferences but relatively few mid-rank ones; a key signature of polarization. This is confirmed in the rerun results as they both win under PL but both lose under BC; see table 4. Candidate 2 however may be described as more consensual than polarized as this person gained a much larger share of the higher preferences ($\sum_{i=1}^{5} T_{Pi}$) and a much smaller share of the lower ones ($\sum_{i=6}^{9} T_{Pi}$) relative to the two polarized candidates. Attracting above-average support across a broad range of high-rank preferences is a hallmark of consensus. Again, this is borne out in the rerun outcomes as candidate 2 wins outright under BC but loses under PL; see table 4.

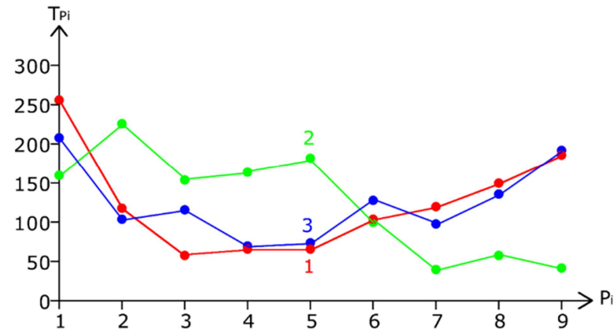

Figure 10. *Preference tallies awarded to candidates 1, 2 and 3.*

As signaled by the consensus indices in table 4, the consensus candidate 2 is promoted in rank and the polarized candidate 3 is demoted as the bias of the vector shifts away from polarization. Note too that candidate 4 is only displaced from a winning rank position under the essentially balanced CHPV or Dowdall vector. In contrast, a candidate in the Anetan constituency only advances to a winning second place under these same rules. So, a balanced vector may promote or demote a candidate's rank position in relation to either PL or BC depending on the type of support awarded to its competitors.

Regarding system bias, both the CHPV and Dowdall vectors are intermediate between BC and PL; see figure 9. In this case study, the winners and the resultant candidate rankings using CHPV are identical to those in all eight

Dowdall elections with just two exceptions. In Aiwo and Ubenide, the two losing candidates ranked immediately above bottom place are reverse-ranked. However, in Boe, the two winning ones are reverse-ranked in the 2022 general election [10]. Here, CHPV and Dowdall differ as to first place. So, is there a justification for choosing one of these two vectors over the other? In other words, is a harmonic or geometric progression of weightings preferable?

In Nauru, Dowdall candidate tallies are rounded to three decimal places. As shown in section 8, CHPV tallies never require rounding so full accuracy is always achievable. If the primary desire is to employ a balanced voting system with a consensus index of ideally 0.5, then CHPV has further advantages over Dowdall. Except for the eight-candidate contests, the CI for CHPV is always closer to 0.5 than Dowdall. Also, when compared to Dowdall, this CHPV index varies much less as a function of N where, as in these elections, $N > 4$. This is important since the choice of vector is normally selected, as here, before the number of candidates is known.

As regards strategic candidate nominations, Dowdall and CHPV perform quite differently. Such nominations occur when one or more 'clone' candidates are added to the ballot; typically for the purpose of manipulating the outcome. In PL ≡ GV(0), adding a clone may take first preferences away from another strong candidate such that their common opponent wins instead. This is called vote-splitting. In contrast, with BC ≡ GV(→1), adding a clone of another candidate may defeat their common opponent by pushing them down in rank position to reduce their tally. This is the other extreme and is known as teaming. The polarized GV(0) vector is vulnerable to vote-splitting but not teaming while the reverse is true for the consensual GV(→1) vector. In smoothly progressing the spectrum from GV(0) to GV(→1), the effect of vote-splitting diminishes while that of teaming grows since this progression is accompanied with the vector's CI ranging from zero towards unity.

For CHPV, by adding K clone candidates, N is increased by K; thereby making this vector more balanced and the likely effect of strategic candidate nominations more uncertain and riskier. For Dowdall, such an addition may entail more than eight candidates in total. Every extra candidate added beyond $N = 8$ makes this vector ever more unbalanced, consensual, and increasingly prone to teaming as its CI heads towards unity. CHPV has the advantage over the Dowdall method in respect of the above four properties.

10. Conclusions

The set of valid PV vectors for an election with N candidates is N-dimensional. Even if attention is solely confined to the unique normalized member of each affine equivalence class, the resulting space is still $(N-2)$-dimensional. According to Saari, this "curse of dimensionality" is "the primary cause for all voting paradoxes" [11]. He also exhorts researchers to "Find appropriate properties and relationships to eliminate subjectivity in selecting a rule" [11]. Given the vast array of available PV vectors from which to choose, one basis on which to do so is the desired bias of the electoral system towards certain types of candidates over others. The bias indices developed here facilitate this choice.

The plurality and Borda count methods are well established. The consensus index for the plurality vector is zero while it tends towards unity - the opposite extreme - for a Borda count election. The Geometric Voting vectors that evolved from the derivation of the bias indices allow the spectrum between these two extremes to be explored. Using the common ratio as the sole variable, this system bias spectrum can be smoothly interpolated for any given number of preferences. GV(r) vectors are hence valuable analytical tools.

Plurality is biased in favor of candidates with strong first-preference support whereas the Borda count instead favors ones that attract broad high-ranking support. Selecting an intermediate vector where neither strong nor broad support dominates the other may seem like an arbitrary choice. However, using GV vector analysis and requiring a bias favoring neither extreme, the Consecutively Halved Positional Voting vector is established as the optimum balanced one; especially when the number of candidates standing is unknown prior to vector selection or large after nominations close.

CHPV is a rival to the Dowdall method that also uses an intermediate vector; one that employs a harmonic - not geometric - progression of weightings. Unlike Dowdall, no rounding need occur in a CHPV count. Also, for the CHPV vector, its bias index is normally closer to the ideal of one half and it varies less than for Dowdall as more candidates beyond four are nominated. Fourthly, CHPV performs differently to Dowdall regarding strategic candidate nominations. Adding additional candidates makes the Dowdall vector more consensual and, hence more likely, increasingly prone to teaming whereas it makes the CHPV one more balanced. This balance may extend to the conflicting effects of vote-splitting that afflicts polarized elections and teaming that afflicts consensus ones. As a vector that is intermediate between these two extremes, CHPV may counterbalance these two opposing effects thus discouraging cloning attempts. This is a major area for further study. With reference to the Gibbard-Satterthwaite theorem on strategic voting, Saari highlights the subsequent search "for incentives and strategy-proof mechanisms, which encourage sincere reactions" [12]. This is another area for more research as CHPV, being a balanced vector, may discourage strategic voting more so than unbalanced ones.

In comparison to plurality, anti-plurality, the Borda count or the Dowdall method as surveyed in a variety of sources [2, 5, 13-15], CHPV is a viable and balanced alternative positional voting system.

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